JOURNAL OF APPROXIMATION THEORY 70, 1–15 (1992)

The Sampling Theorem, L_q^T-Approximation and ϵ-Dimension

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Communicated by P. L. Butzer

Received October 24, 1990; revised April 3, 1991

1. INTRODUCTION

The well-known Whittaker–Kotelnikov–Shannon sampling theorem states that every signal function f which is bandlimited to $[-\sigma, \sigma]$ can be completely reconstructed from its sampled values $f(\pi k/\sigma)$, $k \in Z$. This theorem has several forms. We formulate one of them which was proved by Kotelnikov [14].

THEOREM A. *Let $f \in L_2(R) \cap C(R)$ and the support of the Fourier transform of f be contained in $[-\sigma, \sigma]$. Then for any $x \in R$*

$$f(x) = \sum_{k \in Z} f(\pi k/\sigma) \operatorname{sinc}(\sigma(x - \pi k/\sigma)),$$

where $\operatorname{sinc}(x) = x^{-1} \sin x$ *for* $x \neq 0$ *and* $= 1$ *for* $x = 0$.

Kotelnikov [14] first discovered the information sense of Theorem A. He noted that the quantity of information necessary for reconstructing in the time interval $[-T, T]$ the signal function f defined in this theorem is asymptotically equal to the quantity of information for determining $2\sigma T/\pi$ real numbers. Shannon [18] had a similar idea for random processes.

Let $W \subset C(R)$. Denote by $\mathscr{H}_\varepsilon^T(W)$ the ε-entropy of W in the space $C([-T, T])$. Taking the basic idea of Kotelnikov and Shannon Kolmogorov [13] introduced the superior and inferior ε-entropies per unit length $H_\varepsilon^s(W)$ and $H_\varepsilon^i(W)$ as the superior and inferior limits of $(2T)^{-1} \mathscr{H}_\varepsilon^T(W)$. Tikhomirov [21] obtained the first results on these quantities, in particular, the following precise assertion of Kotelnikov's hypothesis:

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THEOREM B. *Let SB_σ be the set of functions which are bandlimited to $[-\sigma, \sigma]$ and bounded on R by constant* 1. *Then*

$$\lim_{\varepsilon \to 0} H_\varepsilon^s(SB_\sigma)/\log \frac{1}{\varepsilon} = \lim_{\varepsilon \to 0} H_\varepsilon^i(SB_\sigma)/\log \frac{1}{\varepsilon} = \sigma/\pi.$$

Din' Zung [10] proved a generalization of Theorem B in terms of the mean ε-entropy in the space $L_q(R^n)$, $1 \leqslant q \leqslant \infty$, for the set of multivariate functions bandlimited to arbitrary Jordan-measurable subsets of R^n. Latter on, Tikhomirov introduced the mean ε-dimension in $L_q(R^n)$ also based on Kotelnikov and Shannon's idea, but the role of the ε-entropy is replaced by the ε-dimension which is inverse to well-known n-width. The above mentioned approximation characteristics were studied in [8–11, 13, 15, 21] for sets of bandlimited functions, of analytic functions, and of smooth functions of one and several variables. In the study of these quantities the truncation L_q^T-approximation by finite sampling sums played a central role, where $L_q^T = L_q([-T, T]^n)$, $1 \leqslant q \leqslant \infty$.

Also, numerous articles were devoted to various aspects of the classical sampling theorem (Theorem A). Among them the papers of Buslaev and Vituškin [4], Sofman [19, 20], Butzer, Splettstösser, Stens and others from the Aachen group (see bibliography in [5, 6]) Boas [2], Brown [3], Landau, Pollak, Slepian, and Barceló and Córdoba (see bibliography in [1]) are the closest to our paper.

The ε-entropy per unit length, mean ε-entropy, and mean ε-dimension are suitable for expression of the corresponding methods of approximation for functions on R^n only in the case when the ε-entropy or ε-dimension of the set W^T of restrictions on the set Q_T of functions $f \in W$ is asymptotically proportional to the volume of Q_T for T large enough, where $Q_T = \{x \in R^n : |x_i| \leqslant T_i, \quad i = 1, ..., n\}$, $x = (x_1, ..., x_n)$, $T = (T_1, ..., T_n)$, $T_i > 0$. However, this property does not hold, in general, for many cases. In our paper we deal with some such cases. Namely, we investigate the ε-dimension and ε-entropy of the L_p-bounded set $SB_{G,p}$ of multivariate functions bandlimited to the given Jordan-measurable subset G of R^n for various pairs p and q. It turns out that these quantities have asymptotic orders of the form $(\mathrm{vol}\, Q_T)^s F(\varepsilon, p, q)$, $s > 0$, for $T \to \infty$. Here the power s is a function of p and q and can be smaller than 1, equal to 1, or greater than 1. We shall compute s and $F(\varepsilon, p, q)$ for various pairs p and q.

To establish these asymptotic orders we shall present a multivariate modification of the classical sampling theorem, an analogue of Marcinkiewcz' theorem on equivalence between the L_p-norm and the discrete l_p^s-norm for bandlimited functions and the truncation L_q^T-approximation by finite sampling sums for bandlimited functions.

2. PRELIMINARIES

2.1. *Bandlimited Functions*

Let G be a subset of R^n. As usual, we denote by $L_q(G)$, $1 \leqslant q \leqslant \infty$, the normed linear space of all those functions on G for which the integral norm

$$\|f\|_{L_q(G)} = \left(\int_G |f(x)|^q \, dx \right)^{1/q}$$

(with the change to ess supp-norm when $q = \infty$) is finite. If $G = R^n$, then $L_q(G) = L_q$, $\|f\|_{L_q} = \|f\|_q$, and if $G = Q_T$, then $L_q(G) = L_q^T$, $\|f\|_{L_q^T} = \|f\|_{q,T}$.

The locally integrable function f on R^n is said to be bandlimited to G if the support of $\hat{f}$ is contained in G, where $\hat{f}$ denotes the Fourier transform in the distributional sense of f. Denote by $B_{G,p}$ the set of all those functions from L_p which are bandlimited to G. If $G = Q_\sigma$, $\sigma \in R_+^n$, then $B_{G,p} = B_{\sigma,p}$. Here $R_+^n = \{x \in R^n : x_i > 0,\ i = 1, ..., n\}$, and x_i is the ith coordinate of x, i.e., $x = (x_1, ..., x_n)$. The Schwartz theorem states that $B_{\sigma,p}$ coincides with the set of all those functions from L_p which can be continued analytically to entire functions of exponential type $\leqslant \sigma$ (cf. [23]). In harmonic approximation of functions on R^n bandlimited functions play a basic role as trigonometric polynomials for periodic functions.

Let $xy = (x_1 y_1, ..., x_n y_n)$, $\pi/x = (\pi/x_1, ..., \pi/x_n)$, $x^a = \prod_{i=1}^n x_i^a$; in particular, $x^1 = \prod_{i=1}^n x_i^1$ for $x \in R_+^n$, $y \in R^n$, and $a \in R$. The symbol $k \in Z^n$ will be dropped in the series $\sum_{k \in Z^n}$. For the sequence of real or complex numbers $\{a_k\}_{k \in Z^n}$ we define the series-norm $\|\{a_k\}\|_p$, $1 \leqslant p \leqslant \infty$ (with the change to the sup-norm when $p = \infty$),

$$\|\{a_k\}\|_p = \left(\sum |a_k|^p \right)^{1/p}.$$

From the Jackson–Nikolskii inequality

$$\|f\|_q \leqslant 2^n \sigma^{1/p - 1/q} \|f\|_p, \qquad f \in B_{\sigma,p} \quad (1 \leqslant p < q \leqslant \infty) \tag{1}$$

(cf. [17]) follows

$$B_{\sigma,p} \subset B_{\sigma,q} \qquad (1 \leqslant p < q \leqslant \infty). \tag{2}$$

For the following inequality see also [17]:

$$\sup_{x \in R^n} \|\{f(x - hk)\}\|_p \leqslant \prod_{i=1}^n (1 + h_i \sigma_i)\, h_i^{-1/p} \|f\|_p, f \in B_{\sigma,p} (h \in R_+^n). \tag{3}$$

2.2. *Sampling Representations*

The sampling representation in Theorem A has a natural multidimensional generalization. However, it is not suitable in many problems of harmonic approximation. The reason is the slow rate of convergence of the kernel sinc in infinity. There are other more satisfactory sampling representations. The following Cartwright representation was employed [13, 9, 10, 15] for study of the ε-entropy per unit length and mean ε-dimension.

The multidimensional sinc-function is defined by

$$\operatorname{sinc}_n(x) = \prod_{i=1}^{n} \operatorname{sinc}(x_i), \qquad x \in R^n.$$

Let $\sigma, \delta \in R^n_+$ and $\rho = \sigma + \delta$. Put

$$C(x) = \operatorname{sinc}_n(\rho x) \operatorname{sinc}_n(\delta x).$$

For every function $f \in B_{\sigma, p}$ the following sampling representation holds:

$$f(x) = \sum f(hk)\, C(x - hk) \qquad (h = \pi/\rho). \tag{4}$$

The convergence of the series and the equality in (4) are understood pointwise. This formula was proved in [7] for univariate functions and in [9] for multivariate functions.

We need a generalization of the sampling representation (4) which can be proved in a similar way:

THEOREM 1. *Let $1 \leqslant p \leqslant \infty$, $\sigma, \delta \in R^n_+$, $\rho = \sigma + \delta$. Let $\psi \in B_{\delta,2}$ be a function such that $\psi(0) = 1$ and $\varphi(x) = \operatorname{sinc}_n(\rho x)\, \psi(x)$, $x \in R^n$. Then every $f = B_{\sigma, p}$ can be represented by the series*

$$f(x) = \sum f(hk)\, \varphi(x - hk), \qquad x \in R^n \quad (h = \pi/\rho), \tag{5}$$

converging pointwise.

Remark. Theorem 1 can be obtained directly from the Poisson summation formula. By use of (3) one can verify that the series (5) converges uniformly on R^n for $p < \infty$ and uniformly on any compact subset of R^n for $p = \infty$.

Let φ be the function defined in Theorem 1 and let $E_p(\varphi)$ be the set of all those functions from L_p which can be represented by the series

$$f = \sum f_k\, \varphi(\cdot - hk) \qquad (h = \pi/\rho), \tag{6}$$

where the convergence and equality are understood in L_p. From (1) one easily see that the series (6) converges at every $x \in R^n$ to $f(x)$. Moreover, we have, since $f_k = f(hk)$, $k \in Z^n$, for every $f \in E_p(\varphi)$

$$f(x) = \sum f(hk)\, \varphi(x - hk). \tag{7}$$

LEMMA 1. *Under the hypotheses of Theorem* 1 *the following inclusions hold*:

$$B_{\sigma, p} \subset E_p(\varphi) \subset B_{\rho + \delta, p}.$$

Proof. The first inclusion follows from Theorem 1 and (3). It is not hard to check that every finite sum of the series (7) belongs to $B_{\rho + \delta, p}$. Since $B_{\rho + \delta, p}$ is a closed subspace of L_p (cf. [17]) we obtain the second inclusion.

2.3. *An Analogue of Marcinkiewicz' Theorem*

A well-known theorem of Marcinkiewicz establishes the equivalence of the L_p-norm of trigonometric polynomials of order $\leqslant m$ and the discrete l_p^{2m+1}-norm constituted from their values at a uniform lattice (cf. [24]). We prove an analogue of this theorem for bandlimited functions.

THEOREM 2. *Let* $1 \leqslant p \leqslant \infty$, $\sigma, \delta \in R^n_+$, *and* $h = \pi/(\sigma + \delta)$. *Suppose that* σ *and* δ *satisfy the condition* $a\delta_i \leqslant \sigma_i \leqslant b\delta_i$, $i = 1, ..., n$, *for some positive constants* a *and* b. *Then there exist positive constants* $c = c(a, b, p)$ *and* $c' = c'(a, b)$ *such that for every* $f \in B_{\sigma, p}$

$$c\sigma^{1/p}\, \|f\|_p \leqslant \|\{f(hk)\}\|_p \leqslant c'\sigma^{1/p}\, \|f\|_p. \tag{8}$$

Proof. We prove the theorem for the case $1 < p < \infty$. The cases $p = 1, \infty$ can be proved in a similar way. The second inequality of (8) is a simple consequence of (3). Let $f \in B_{\sigma, p}$. Consider the sampling representation (5) of f with $\varphi(x) = \operatorname{sinc}_n(\rho x) \operatorname{sinc}_n(\delta x)$, $\rho = \sigma + \delta$. By Hölder's inequality we have

$$|f(x)|^p \leqslant \sum |f(hk) \operatorname{sinc}_n(\rho(x - hk))|^p\, \|\{\operatorname{sinc}_n(\delta(x - hk))\}\|_{p'}^p,$$

where $1/p + 1/p' = 1$. Since $\operatorname{sinc}_n(\delta\, \cdot) \in B_{\delta, p'}$ by (3) one can verify that

$$\|\{\operatorname{sinc}_n(\delta\, \cdot)\}\|_{p'} \leqslant M,$$

where M is a certain positive constant depending on a, b, and p (n fixed).

Combining both inequalities we obtain

$$\|f\|_p^p \leqslant M^p \sum |f(hk)|^p \, \|\mathrm{sinc}_n(\rho\,\cdot)\|_p^p.$$

Hence the first inequality of (8) follows.

3. The Truncation L_q^T-Approximation

Let F and G be non-negative functions defined in the set X. We write $F(x) \ll G(x)$, $x \in X$, if there exists a positive constant c such that $F(x) \leqslant cG(x)$ for every $x \in X$ and write $F(x) \asymp G(x)$, $x \in X$, if $F(x) \ll G(x)$ and $G(x) \ll F(x)$, $x \in X$. We denote $x \leqslant y$ $(x < y)$ for $x, y \in R^n$ if $x_i \leqslant y_i$ $(x_i < y_i)$, $i = 1, ..., n$.

For the function $f \in E_p(\varphi)$ we define the finite sampling sum $S_N f$, $N \in R^n_+$, from the sampling series (7) by

$$(S_N f)(x) = \sum_{k \in Z_N} f(hk)\, \varphi(x - hk),$$

where $Z_N = \{k \in Z^n : |k_i| \leqslant N_i, i = 1, ..., n\}$.

Theorem 3. *Let $1 \leqslant p, q \leqslant \infty$, $\sigma, \delta \in R^n_+$, $m \in N$. Let $\varphi(x) = \mathrm{sinc}_n(\rho x)\{\mathrm{sinc}_n(\delta x/m)\}^m$. Suppose that σ and δ satisfy the condition $a\delta \leqslant \sigma \leqslant b\delta$ for some positive constants a and b. Then*

$$\|f - S_N f\|_{q,T} \ll \sum_I A_I \, \|f\|_p, \qquad f \in E_p(\varphi),\ T, N \in R^n_+,\ Nh > T,$$

where the sum ranges through all proper subsets I of the set J of natural numbers at most n and

$$A_I = \prod_{i \in I} T_i^{1/q}(N_i h_i - T_i)^{-m-r}, \qquad r = \min\left(\frac{1}{p}, \frac{1}{q}\right).$$

Proof. We prove Theorem 3 for the case $1 < p$, $q < \infty$. The remainder of cases can be proved in a similar way. Let us first consider the case $p = q$. Let $f \in E_p(\varphi)$. It is not hard to check the inequality

$$\|f - S_N f\|_{p,T} \ll \sum_I \|f_{I,N}\|_{p,T},$$

where

$$f_{I,N}(x) = \sum_{k \in Z_{I,N}} f(hk)\, \varphi(x - hk),$$

$Z_{I,N}=\{k\in Z^n : |k_i|>N_i,\ i\in I\}$. Thus, to prove the theorem it is sufficient to establish the inequality

$$\|f_{I,N}\|_{p,T}\ll A_I\,\|f\|_p, \qquad f\in E_p(\varphi),\ N,\ T\in R^n_+,\ Nh>T, \tag{9}$$

for an arbitrary subset I of J, $I\neq J$. For the sake of simplicity, we verify it for $I=\{1,2,...,s\}$, $1\leqslant s<n$. Write $x=(x',x'')$, $x'=(x_1,...,x_s)$, $x''=(x_{s+1},...,x_n)$. We have

$$f_{I,N}(x)=\sum_{k'\in P_{N'}} f(h'k',x'')\,\varphi'(x'-h'k'),$$

where $P_{N'}=\{k'\in Z^s : |k_i|>N_i,\quad i=1,...,s\}$ and $\varphi'(x')=\mathrm{sinc}_s(\rho'x')\{\mathrm{sinc}_s(\delta'x'/m)\}^m$. Then, by Hölder's inequality,

$$|f_{I,N}(x)|\leqslant\left(\sum_{k'\in Z^s}|f(h'k',x'')|^p\right)^{1/p}\left(\sum_{k'\in P_{N'}}|\varphi'(x'-h'k')|^{p'}\right)^{1/p'}, \tag{10}$$

where $1/p+1/p'=1$. By an easy estimation we obtain

$$\left\|\left(\sum_{k'\in P_{N'}}|\varphi'(\cdot-h'k')|^{p'}\right)^{1/p'}\right\|_{p,T'}\ll A_I. \tag{11}$$

Since for fixed $x''\in R^{n-s}$ the function $f(\cdot,x'')$ belongs to $E_p(\varphi')$ as a function of the variables $x_1,...,x_s$, by Lemma 1 and (3) we get

$$\|\{f(h'k',x'')\}\|_p\ll\|f(\cdot,x'')\|_p$$

(σ and δ fixed). Hence,

$$\|\ \|\{f(h'k',\cdot)\}\|_p\ \|_p\ll\|f\|_p. \tag{12}$$

Combining (10)–(12) we prove (9). Therefore the theorem is proved for $p=q$. From this case we obtain the theorem for $p>q$ because of the trivial inequality $\|f\|_{q,T}\leqslant(2T)^{1/q-1/p}\,\|f\|_{p,T}$, and for $p<q$ because of the inequality $\|f\|_q\ll\|f\|_p$, $f\in E_p(\varphi)$ which follows from Lemma 1 and (1).

In a similar way we can prove

THEOREM 4. *Under the hypotheses of Theorem* 3 *we have*

$$\|S_Nf\|_{p,T}\geqslant\|S_Nf\|_p\left(1-\sum_I B_I\right), \qquad f\in E_p(\varphi),\ N,\ T\in R^n_+,\ Nh<T,$$

where the sum ranges through all proper subsets I of J and

$$B_I\ll\prod_{i\in I}N_i^{1-1/p}(T_i-N_ih_i)^{1/p-m}, \qquad T,\ N\in R^n_+,\ Nh<T.$$

4. The ε-Dimension and ε-Entropy

4.1. *ε-Dimension*

Let X be a normed linear space and A and B be subsets of X. Set

$$E(A, B, X) = \sup_{x \in A} \inf_{y \in B} \|x - y\|.$$

Denote by $\mathscr{L}_m$ the set of linear manifolds in X of dimensions at most m. The quantity

$$d_m(A, X) = \inf_{L \in \mathscr{L}_m} E(A, L, X)$$

is called the m-dimensional Kolmogorov width of A. The quantity

$$\mathscr{K}_\varepsilon(A, X) = \inf\{m : \exists L \in \mathscr{L}_m : E(A, L, X) \leqslant \varepsilon\} \quad (\varepsilon > 0)$$

is called the ε-dimension of A. The last approximation characteristic, inverse to the width $d_m(A, X)$, expresses the necessary dimensions of a linear manifold for approximation of A within to ε.

Denote by meas G the measure of measurable $G \subset R^n$. For $W \subset L_p$ SW is the intersection of W with the unit ball of L_p and W^T is the set of restrictions of functions of W in the set $Q_T = \{x \in R^n : |x_i| \leqslant T_i,\ i = 1, ..., n\}$. The following theorem was proved in [10].

Theorem C. *Let $1 \leqslant p \leqslant \infty$ and let G be a Jordan-measurable and bounded subset of R^n. Then*

$$\lim_{T \to \infty} (2T)^{-1} \mathscr{K}_\varepsilon((SB_{G,p})^T, L_p^T) = (2\pi)^{-n} \operatorname{meas} G.$$

This theorem shows that the ε-dimension per unit volume of $SB_{G,p}$ is equal asymptotically to $(2\pi)^{-n}$ meas G expressing the bandwidth of functions from $SB_{G,p}$, and, consequently, does not depend on ε when $T \to \infty$.[1] Moreover, one can also see that the necessary dimensions of a linear manifold for the L_p^T-approximation of $SB_{G,p}$ within arbitrary ε is proportional asymptotically to the volume of Q_T as well the bandwidth of functions from $SB_{G,p}$. However, this property does not hold, in general, for the same L_q^T-approximation of $SB_{G,p}$ with various pairs p and q.

First we prove that the ε-dimension per unit volume of $SB_{G,p}$ is not bigger asymptotically than $(2\pi)^{-n}$ meas G for every pair p and q. Later we will see that this quantity can be equal asymptotically to zero for the case $q > 2$ (Theorem 7). Set $\mathscr{K}_\varepsilon(G, T) = \mathscr{K}_\varepsilon((SB_{G,p})^T, L_q^T)$ for the fixed pair p and q.

[1] $T \to \infty$ means $T_i \to +\infty$ for $i = 1, ..., n$.

THEOREM 5. *Let $1 \leqslant p, q \leqslant \infty$ and let G be a Jordan-measurable and bounded subset of R^n. Then*

$$\limsup_{T\to\infty} (2T)^{-1} \mathscr{K}_\varepsilon(G, T) \leqslant (2\pi)^{-n} \operatorname{meas} G \quad (0<\varepsilon<1).$$

Proof. *Step* 1. Let $G=Q_6$, i.e., $B_{G,p}=B_{\sigma,p}$. Taking an arbirary $\delta \in R^n_+$, by Theorem 1 we have for every $f \in B_{G,p}$

$$f(x)=\sum f(hk)\,\varphi(x-hk),$$

where $\varphi(x)=\operatorname{sinc}_n(\rho x)\{\operatorname{sinc}_n(\delta x/2)\}^2$, $\rho=\delta+\sigma$, $h=\pi/\rho$. Let α be a fixed positive number, $\frac{1}{2}<\alpha<1$. We define $N \in R^n_+$ by $N_i h_i = T_i + T_i^\alpha$, $i=1, ..., n$, for $T \in R^n_+$. From Theorem 3 it is not hard to check that $\|f-S_N f\|_{q,T}$ converges to 0 uniformly on $SB_{\sigma,p}$, as $T\to\infty$. Thus, for arbirary $0<\varepsilon<1$, there exists some $T^0 \in R^n_+$ such that $\|f-S_N f\|_{q,T} \leqslant \varepsilon$ for all $T \geqslant T^0$ and $f \in SB_{\sigma,p}$. Hence,

$$E((SB_{\sigma,p})^T, L^T, L_q^T) \leqslant \varepsilon,$$

where L is the linear hull of the functions $\varphi(\cdot - hk)$, $k \in Z_N$. This implies that

$$\mathscr{K}_\varepsilon(G, T) \leqslant \prod_{i=1}^{n} (2N_i+1) = \pi^{-n}(2T)^1\,\varphi^1 + o(T^1)$$

because $\dim L \leqslant \prod_{i=1}^n (2N_i+1)$. Letting $T\to\infty$, we obtain

$$\limsup_{T\to\infty} (2T)^{-1} \mathscr{K}_\varepsilon(G, T) \leqslant \pi^{-n}(\sigma+\delta)^1.$$

As δ is arbitrary, this proves the theorem for the case $G=Q_\sigma$.

Step 2. Let $G=\bigcup_{j=1}^m (x^j+Q_\sigma)$ and $\operatorname{int}(x^j+Q_\sigma) \cap \operatorname{int}(x^{j'}+Q_\sigma) \neq \varnothing$, for $j \neq j'$. This case can be proved in a way similar to Step 1 using the representation for $f \in SB_{G,p}$

$$f(x)=\sum_{j=1}^{m} \exp(i\langle x^j, x\rangle)\, f_j(x),$$

where $f_j \in B_{\sigma,p}$ and $\|f_j\|_p \leqslant c$, the constant c does not depend on f (cf. [9]).

Step 3. Let G be an arbitrary Jordan-measurable set. We will follow [9]. There exist sets G' and G'' of the form considered in Step 2 such that $G' \subset G \subset G''$ and the measure of $G' \setminus G''$ is as small as we like. Now the theorem follows from the case considered in Step 2 and from the trivial inclusions $SB_{G',p} \subset SB_{G,p} \subset SB_{G'',p}$.

The following two theorems sharpen Theorem 5.

THEOREM 6. *Under the hypotheses of Theorem 5 let $p \geqslant q$. Then for some $0 < \varepsilon_0 < 1$*

$$\lim_{T \to \infty} (2T)^{-1} \mathscr{K}_\varepsilon(G, T) = (2\pi)^{-n} \operatorname{meas} G \qquad (0 < \varepsilon \leqslant \varepsilon_0).$$

Proof. In virtue of Theorem C we must prove the theorem for the case $p > q$. For the latter case from Theorem C and (2) we can easily verify the inequality

$$\liminf_{T \to \infty} (2T)^{-1} \mathscr{K}_\varepsilon(G, T) \geqslant (2\pi)^{-n} \operatorname{meas} G \qquad (0 < \varepsilon < \varepsilon_0)$$

for some $0 < \varepsilon_0 < 1$. This and Theorem 5 imply Theorem 6 for $p > q$.

Denote by B_p^s, $1 \leqslant p \leqslant \infty$, the unit ball of the normed linear space l_p^s of finite sequences $\{x_k\}_{k=1}^s$ with the norm (this norm is changed to max-norm when $p = \infty$)

$$\|\{x_k\}\|_{l_p^s} = \left(\sum_{k=1}^{s} |x_k|^p \right)^{1/p}.$$

THEOREM 7. *Under the hypotheses of Theorem 5 let $p < q$ and $G = Q_\sigma$. Then the following assertions hold*:

(i) *For $q < 2$*

$$\limsup_{T \to \infty} (2T)^{-1} \mathscr{K}_\varepsilon(G, T) \asymp \liminf_{T \to \infty} (2T)^{-1} \mathscr{K}_\varepsilon(G, T) \asymp \sigma^1, \ \sigma \in R_+^n, \ 0 < \varepsilon < 1.$$

(ii) *For $q > 2$*

$$\limsup_{T \to \infty} (2T)^{-2/q} \mathscr{K}_\varepsilon(G, T) \asymp \liminf_{T \to \infty} (2T)^{-2/q} \mathscr{K}_\varepsilon(G, T) \asymp \varepsilon^{-\nu} \sigma^\mu, \ \sigma \in R_+^n, \ 0 < \varepsilon \leqslant \varepsilon_0,$$

where $0 < \varepsilon_0 < 1$, $\nu = 2 \max\{1, (1/2 - 1/q)/(1/p - 1/q)\}$ and $\mu = \max\{1, 2/p\}$.

Proof. We prove the theorem in the case $p < 2 < q$. The other cases can be proved in a similar way.

Upper Estimate. For $p < 2 < q$ we check the inequality

$$\limsup_{T \to \infty} (2T)^{-2/q} \mathscr{K}_\varepsilon(G, T) \ll \varepsilon^{-2} \sigma^{2/p}, \qquad \sigma \in R_+^n, \ 0 < \varepsilon \leqslant \varepsilon_0. \tag{13}$$

Consider the sampling representation (5) of $f \in B_{G,p}$ with $\varphi(x) = \operatorname{sinc}_n(2\sigma x)\{\operatorname{sinc}_n(\sigma x/2)\}^2$, $\delta = \sigma$, $\rho = 2\sigma$, $h = \pi/2\sigma$. Define $N \in R^n_+$ by $Nh = 2T$ for $T \in R^n_+$ and put $V = \{S_N f : f \in SB_{G,p}\}$ and $U = \{f - S_N f : f \in SB_{G,p}\}$, where $S_N f$ is the associated finite sampling sum. In view of the inclusion $SB_{G,p} \subset V + U$, we have

$$\mathscr{K}_\varepsilon(G, T) \leqslant \mathscr{K}_{\varepsilon/2}(V^T, L^T_q) + \mathscr{K}_{\varepsilon/2}(U^T, L^T_q). \tag{14}$$

First we estimate the first term in the right-hand part of (14). By Theorem 2 we get

$$\|\{f(hk)\}\|_{l^s_p} \ll \sigma^{1/p} \|f\|_p$$

and

$$\|\{f(hk)\}\|_{l^s_q} \gg \sigma^{1/q} \|f\|_{q,T}, \qquad f \in V,$$

where $s = \prod_{i=1}^n (2[N_i] + 1)$ (here and hereafter $[a]$ denotes the integral part of $a \in R$). Hence by properties of the m-width one can verify that for $m < s$

$$d_m(V^T, L^T_q) \leqslant c_1 \sigma^{1/p - 1/q} d_m(B^s_p, l^s_q), \tag{15}$$

where $c_1 = c_1(p, q)$ is a positive constant. Set $m = (\lambda\varepsilon)^{-2} (2T)^{2/q} \sigma^{2/p}$, where the constant will be defined later. Applying the estimate of $d_m(B^s_p, l^s_q)$ in [12], we obtain

$$\begin{aligned} d_m(B^s_p, l^s_q) &\ll s^{1/q} m^{-1/2} \\ &\leqslant c_2 \lambda\varepsilon\sigma^{1/q - 1/p}, \qquad \sigma \in R^n_+, 0 < \varepsilon < \varepsilon_0, \end{aligned} \tag{16}$$

where $\varepsilon_0 = \varepsilon_0(\lambda)$, $c_2 = c_2(p, q)$ are positive constants. Defining $\lambda = (2c_1c_2)^{-1}$, from (15) and (16) we have $d_m(V^T, L^T_q) \leqslant \varepsilon/2$. This implies

$$\mathscr{K}_{\varepsilon/2}(V^T, L^T_q) \leqslant m \ll \varepsilon^{-2} \sigma^{2/p} T^{2/q}, \qquad \sigma, T \in R^n_+, 0 < \varepsilon < \varepsilon_0. \tag{17}$$

We now estimate the second term in the right-hand part of (14). From Theorem 3 it follows that $\|f - S_N f\|_{q,T}$ converges to zero uniformly on $SB_{G,p}$ as $T \to \infty$. Hence we obtain the equality $\mathscr{K}_{\varepsilon/2}(U^T, L^T_q) = 0$ (here σ and ε are fixed). This and (14), (17) prove (13).

Lower Estimate. From (13) we can see that the theorem will be proved for the case $p < 2 < q$ if the following relation is true:

$$\begin{aligned} \liminf_{T \to \infty} & (2T)^{-2/q} \mathscr{K}_\varepsilon(G, T) \\ & \gg \varepsilon^{-2} \sigma^{2/p}, \qquad \sigma \in R^n_+, 0 < \varepsilon < \varepsilon_0. \end{aligned} \tag{18}$$

Let $\varphi'(x) = \{\mathrm{sinc}_n(\sigma x/2)\}^2$ and let α be a fixed positive number, $0 < \alpha < \frac{1}{2}$. We define $N \in R^n_+$ by $N_i h_i = T_i - T_i^\alpha$, $i = 1, ..., n$, where $h = 2\pi/\sigma$. Let H be the set of all linear combinations f of the functions $\varphi'(\cdot - hk)$, $k \in Z_N$, such that $\|f\|_p \leqslant 1$. In order to prove (18), since by Lemma 1 $H \subset SB_{G,p}$, it is sufficient to check the inequality

$$\liminf_{T \to \infty} (2T)^{-2/q} \mathscr{H}_\varepsilon(H^T, L_q^T) \geqslant \varepsilon^{-2} \sigma^{2/p}, \qquad \sigma \in R^n_+, 0 < \varepsilon < \varepsilon_0. \tag{19}$$

From Theorem 4 it is not hard to verify that $\|f\|_{p,T}$ converges to $\|f\|_p$ uniformly on H when $T \to \infty$. Hence by Theorem 2 we have

$$\|f\|_{t,T} \asymp \sigma^{-1/t} \|\{f(hk)\}\|_{l_t^r}, \qquad f \in H, T \geqslant T^0 \quad (t = p, q),$$

where $r = \prod_{i=1}^n (2[N_i] + 1)$. By the latter relation, analogously to that in the upper estimate, one can see that

$$d_m(H^T, L_q^T) \geqslant c_3 \sigma^{1/p - 1/q} d_m(B_p^r, l_q^r),$$

where $c_3 = c_3(p, q)$ is a positive constant. Further, the proof of (19) can be continued in the same way as in the proof of (13).

4.2. *ε-Entropy*

Let X be a metric space and A be a compact subset of X. Denote by $\mathscr{N}_\varepsilon(A, X)$, $\varepsilon > 0$, the minimal number of elements of an ε-net of A. Then the quantity $\mathscr{H}_\varepsilon(A, X) = \log \mathscr{N}_\varepsilon(A, X)$ is called the ε-entropy of A (cf. [13]). This quantity expresses the necessary number of bits for the binary recording the "information" set A within to ε.

LEMMA 2. *Let X be a Banach space and A be a compact subset of X. Then*

$$\mathscr{H}_\varepsilon(A, X) \leqslant \mathscr{K}_{\varepsilon/2}(A, X) \log\{8(d_0 + \varepsilon)/\varepsilon\},$$

where $d_0 = d_0(A, X)$.

Proof. Let $m(y) = \sup\{m : d_m(A, X) \geqslant 1/y\}$. The lemma follows from the trivial inequality $\mathscr{K}_\varepsilon(A, X) \geqslant m(1/\varepsilon)$ and the inequality $m(2/\varepsilon) \log\{8(d_0 + \varepsilon)/\varepsilon\} \geqslant \mathscr{H}_\varepsilon(A, X)$, proved in [16].

LEMMA 3. *Let G be a bounded subset of R^n. Then $(SB_{G,p})^T$ is a compact subset of L_q^T for any pair p, q and $T \in R^n_+$.*

Proof. It is clear that we must prove the lemma only for $G = Q_\sigma$, and, because of (2), for $p = \infty$. This case was proved by Bernstein and Nikolskii for $p=q=\infty$ (cf. [17]). Hence by the inequality $\|f\|_{q,T}\leqslant(2T)^{1/q}\|f\|_{\infty,T}$ the lemma follows for $p=\infty$, $q<\infty$.

Set $\mathscr{H}_\varepsilon(G,T)=\mathscr{H}_\varepsilon((SB_{G,p})^T, L_q^T)$ for the fixed pair p and q.

THEOREM 8. *Let* $1\leqslant p=q\leqslant\infty$ *and let* G *be a Jordan-measurable bounded subset of* R^n. *Then*

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\log^{-1}1/\varepsilon(2T)^{-1}\mathscr{H}_\varepsilon(G,T) \tag{20}$$

$$=\lim_{\varepsilon\to 0}\liminf_{T\to\infty}\log^{-1}1/\varepsilon(2T)^{-1}\mathscr{H}_\varepsilon(G,T)$$

$$=(2\pi)^{-n}\operatorname{meas}G. \tag{21}$$

Proof. Since $(SB_{G,p})^T$ is contained in the unit ball of L_p^T, we have $d_0((SB_{G,p})^T, L_p^T)\leqslant 1$. Hence and from Lemmas 2 and 3 and Theorem 5 it follows that (20) is not greater than $(2\pi)^{-n}\operatorname{meas}G$. Thus, the theorem will be proved if we show that the left-hand part of (21) is not less than $(2\pi)^{-n}\operatorname{meas}G$. As in estimates for ε-dimension in Theorem 5 we verify this inequality only for the case $G=Q_\sigma$. Let $\varphi(x)=\operatorname{sinc}_n(\rho x)\operatorname{sinc}_n(\delta x)$, $\delta<\sigma$, $\rho=\sigma-\delta$. Taking a fixed positive number α, $0<\alpha<\frac{1}{2}$, we define $N\in R^n_+$ by $N_ih_i=T_i-T_i^\alpha$, where $h=\pi/\rho$. Let F be the set of linear combinations f of the functions $\varphi(\cdot-hk)$, $k\in Z_N$, such that $\|f\|_p\leqslant 1$. Using Theorem 4 one can check that there exists $T^0\in R^n_+$ such that $2^{-1}F(T)\subset F^T$ for any $T\geqslant T^0$, where $F(T)=\{f\in F^T:\|f\|_{p,T}\leqslant 1\}$. Applying the estimate for the ε-entropy of finite dimensional sets in [22, pp. 276–279] we obtain

$$\mathscr{H}_\varepsilon(2^{-1}F(T), L_p^T)\geqslant s\log\tfrac{1}{4}=\pi^{-n}(2T)^1(\sigma-\delta)^1+o(T^1)$$

as $T\to\infty$, where $s=\prod_{i=1}^n(2[N_i]+1)$ is the dimension of the linear hull of $F(T)$. Hence, since δ is arbitrary and by Lemma 1 $2^{-1}F(T)\subset F^T\subset(SB_{G,p})^T$, it is easy to verify that the left-hand part of (21) is not less than $\pi^{-n}\sigma^1=(2\pi)^{-n}\operatorname{meas}G$.

Employing Theorem 7 in a similar way we obtain the following

THEOREM 9. *Under the hypotheses and notation of Theorem* 7 *the following assertions hold*:

(i) *For* $q<2$

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\log^{-1}1/\varepsilon(2T)^{-1}\mathscr{H}_\varepsilon(G,T)\asymp\sigma^1,\qquad \sigma\in R^n_+.$$

(ii) *For* $q>2$

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\varepsilon^{-\nu}\log^{-1}1/\varepsilon(2T)^{-2/q}\mathscr{H}_\varepsilon(G,T)\asymp\sigma^\mu,\ \sigma\in R^n_+.$$

Remark. Since the unit ball is Jordan-measurable (a set G is called Jordan-measurable if the Riemann integral of the characteristic function of G exists), Theorems 5, 6 and 8 and, as is easy to see, Theorems 7(i) and 8(i) are true in the case G in $B_{G,p}$ is the unit ball.

ACKNOWLEDGMENTS

I thank Professor Dr. P. L. Butzer who kindly invited me to Aachen for a Colloquium talk, where I had the opportunity to speak about the results of this paper, for valuable discussions and help in making contacts with German specialists on approximation.

I also thank a referee for his very constructive report.

REFERENCES

1. J. A. BARCELÓ AND A. J. CÓRDOBA, Band-limited functions: L^p-convergence, *Bull. Amer. Math. Soc.* **18** (1988), 163–166.
2. R. P. BOAS, Summation formulas and band-limited signals, *Tôhoku Math. J.* **24** (1972), 121–125.
3. J. L. BROWN JR., On the error in reconstructing a non-bandlimited function by means of the bandpass sampling theorem, *J. Math. Anal. Appl.* **18** (1967), 75–84.
4. V. I. BUSLAEV AND A. G. VITUŠKIN, An estimate of the length of a signal code with a finite spectrum in connection with sound transcription problem, *Izv. Akad. Nauk SSSR, Ser. Mat.* **38** (1975), 867–875.
5. P. L. BUTZER, W. SPLETTSTÖSSER, AND R. L. STENS, The sampling theorem and linear prediction in signal analysis, *Jahresber. Deutsch. Math.-Verein.* **90** (1988), 1–70.
6. P. L. BUTZER, A survey of the Whittaker–Shannon sampling theorem and some of its extensions, *J. Math. Res. Exposition* **3** (1983), 185–212.
7. M. L. CARTWRIGHT, On certain integral functions of order 1, *Quart. J. Math. Oxford Ser. (1)* **75** (1936), 46–55.
8. DIN' ZUNG, Some approximation characterics of the classes of smooth functions of several variables in the metric of L_2, *Uspekhi Mat. Nauk* **34** (1979), 189–190.
9. DIN' ZUNG, Mean ε-dimension of the functional class $B_{G,p}$, *Mat. Zametki* **28** (1980), 727–736.
10. DIN' ZUNG, "Approximation of Smooth Functions of Several Variables by Means of Harmonic Analysis," Doctor's Dissertation, Moscow State University, 1985.
11. DIN' ZUNG AND G. G. MAGARIL-IL'JAEV, Problems of Bernštein and Favard type and the ε-dimension of some classes of functions, *Dokl. Akad. Nauk SSSR* **249** (1979), 1345–1348.
12. E. D. GLUSKIN, Norms of random matrices and widths of finite dimensional sets, *Mat. Sb.* **120** (1983), 180–189.
13. A. N. KOLMOGOROV AND V. M. TIKHOMIROV, ε-entropy and ε-capacity of sets in function spaces, *Uspekhi Mat. Nauk* **14** (1959), 3–86.
14. V. A. KOTELNIKOV, On the truncation capacity of "ether" and wire in electrocommunications, *in* "Material for the First All-Union Conference on Questions of Communications, Moscow 1933."
15. LE TRUONG TUNG, Mean ε-dimension of the class of functions whose Fourier transform have support contained in a preassigned set, *Vestnik Moscov. Univ. Ser. I Mat. Mech.* **5** (1980), 44–49.

16. B. S. MIT'JAGIN, Connection between the ε-entropy, the approximation speed and the kernel of a compact subset of a linear space, *Dokl. Akad. Nauk SSSR* **134** (1960), 765–759.
17. S. M. NIKOLSKII, "Approximation of Functions of Several Variables and Embedding Theorems," Nauka, Moscow, 1969.
18. C. E. SHANNON, Communication in the presence of noise, *Proc. IRE* **37** (1949), 10–21.
19. L. B. SOFMAN, A problem of Shannon, *Dokl. Akad. Nauk SSSR* **215** (1974), 1313–1316.
20. L. B. SOFMAN, On asymptotic behavior of the δ, ε-entropy and the δ, ε-dimension of a Gauss homogeneous field, *Uspekhi Mat. Nauk* **36** (1981), 191–194.
21. V. M. TIKHOMIROV, On ε-entropy of some classes of analytic functions, *Dokl. Akad. Nauk SSSR* **117** (1957), 191–194.
22. V. M. TIKHOMIROV, "Some Problems in Approximation Theory," Moscow State Univ. Press, Moscow, 1979.
23. K. YOSIDA, "Functional Analysis," Springer-Verlag, Berlin/New York, 1978.
24. A. ZYGMUND, "Trigonometric Series, II," Cambridge Univ. Press, London/New York, 1959.